

Relativistic field theories in a magnetic background as noncommutative field theories

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We study the connection of the dynamics in relativistic field theories in a strong magnetic field with the dynamics of noncommutative field theories (NCFT). As an example, the Nambu-Jona-Lasinio models in spatial dimensions $d \geq 2$ are considered. We show that this connection is rather sophisticated. In fact, the corresponding NCFT are different from the conventional ones considered in the literature. In particular, the UV/IR mixing is absent in these theories. The reason of that is an inner structure (i.e., dynamical form-factors) of neutral composites which plays an important role in providing consistency of the NCFT. An especially interesting case is that for a magnetic field configuration with the maximal number of independent nonzero tensor components. In that case, we show that the NCFT are finite for even d and their dynamics is quasi-(1+1)-dimensional for odd d . For even d , the NCFT describe a confinement dynamics of charged particles. The difference between the dynamics in strong magnetic backgrounds in field theories and that in string theories is briefly discussed.

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I. INTRODUCTION

Recently, there has been a considerable interest in noncommutative field theories (NCFT) (for reviews, see Ref. [1]). Besides being interesting in themselves, noncommutative theories mimic certain dynamics in quantum mechanical models [2, 3], nonrelativistic field systems [4, 5], nonrelativistic magnetohydrodynamical field theory [6], and string theories [7, 8]. In particular, NCFT are intimately related to the dynamics in quantum mechanical and nonrelativistic field systems in a strong magnetic field [2, 3, 4, 5, 6] and, in the case of open strings attached to D -branes, to the dynamics in string theories in magnetic backgrounds [3, 8, 9].

In this paper, we study the connection between the dynamics in relativistic field theories in a strong magnetic field and that in NCFT. Our main conclusion is that although field theories in the regime with the lowest Landau level (LLL) dominance indeed determine a class of NCFT, these NCFT are different from the conventional ones considered in the literature. In particular, the UV/IR mixing, taking place in the conventional NCFT [10], is absent in this case. The reason of that is an inner structure (i.e., dynamical form-factors) of neutral composites in these theories.

In order to be concrete, we will consider the $(d + 1)$ -dimensional Nambu-Jona-Lasinio (NJL) models in a strong magnetic field for arbitrary $d \geq 2$. In the regime with the LLL dominance, we derive the effective action of the corresponding NCFT in the models with a large number of fermion colors N and analyze their dynamics. These NCFT are consistent and quite sophisticated. An especially interesting case is that for a magnetic field configuration with the maximal number of independent nonzero tensor components. In that case, the theories are finite for even d and their dynamics is quasi-(1+1)-dimensional for odd d [for even d , the NCFT describe a confinement dynamics of charged particles]. As will be shown in this paper, it is the LLL dominance that provides the exponentially damping (form-) factors which are responsible for finiteness of these NCFT for even d and their quasi-(1+1)-dimensionality for odd d . Thus, besides being low energy theories of the NJL models in a strong magnetic field, the NCFT based on the LLL dynamics are self-contained and self-consistent.

We will use two different sets of composite fields for the description of the dynamics. The first set uses the conventional composite fields $\sigma(x) \sim \bar{\psi}(x)\psi(x)$ and $\pi(x) \sim i\bar{\psi}(x)\gamma_5\psi(x)$. In this case, besides the usual Moyal factor, additional exponentially damping factors occur in the interaction vertices of the fields $\sigma(x)$ and $\pi(x)$. These factors reflect an inner structure of composites and play an important role in providing consistency of these NCFT. In particular, because of them, the UV/IR mixing is absent in these theories. In the second approach, one considers other, “smeared”, fields $\Sigma(x)$ and $\Pi(x)$, connected with $\sigma(x)$ and $\pi(x)$ through a non-local transformation. Then, while the additional factors are removed in the vertices of the smeared fields, they appear in their propagators, again resulting in the UV/IR mixing removal. By using the Weyl symbols of the smeared fields, we derive the effective

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action for the composites in the noncommutative coordinate space.

The paper is organized as follows. In Section II, in order to understand the nature of the modified NCFT in a clear and simple way, we discuss the quantum mechanical model in a magnetic field introduced in Ref. [3]. We show that besides the solution of Ref. [3], which mimics a conventional NCFT, there is another solution, with an interaction vertex containing exponentially damping factors. The existence of these two solutions reflects the possibility of two different treatments of the case with the particle mass $m \rightarrow 0$ in this model. In Section III, the effective action of the NCFT connected with the (3+1)-dimensional NJL model in a strong magnetic field is derived. In Section IV, the dynamics of this model is discussed. In Section V, we generalize the analysis to a general case of $d + 1$ dimensions with $d \geq 2$. In Section VI, we summarize the main results of the paper. In Appendices A and B, some useful formulas and relations are derived.

II. NONRELATIVISTIC MODEL

In order to understand better the nature of the modified NCFT, in this section we analyze a simple quantum mechanical two-dimensional system: a pair of unit charges of opposite sign (i.e., a dipole) in a constant magnetic field and with a harmonic potential interaction between them. This model was considered in Ref. [3]. It was argued there that for a strong magnetic field this simple system reproduces the dynamics of open strings attached to D-branes in antisymmetric tensor backgrounds.

We will show that important features of the modified NCFT occur already in this simple quantum mechanical model. Its Lagrangian reads

$$L = \frac{m}{2}(\dot{\vec{x}}_1^2 + \dot{\vec{x}}_2^2) + \frac{eB}{2}\epsilon^{ab}(\dot{x}_1^a x_1^b - \dot{x}_2^a x_2^b) - \frac{K}{2}(\vec{x}_1 - \vec{x}_2)^2, \quad \epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (1)$$

It is convenient to use the center of mass and relative coordinates, $\vec{X} = \frac{\vec{x}_1 + \vec{x}_2}{2}$ and $\vec{\Delta} = \frac{\vec{x}_1 - \vec{x}_2}{2}$. In these coordinates, Lagrangian (1) takes the form

$$L = m(\dot{\vec{X}}^2 + \dot{\vec{\Delta}}^2) + 2eB\epsilon^{ab}\dot{X}^a\Delta^b - 2K\vec{\Delta}^2. \quad (2)$$

The LLL dominance occurs when either $B \rightarrow \infty$ or $m \rightarrow 0$. Taking $m = 0$, the authors of [3] drop the kinetic terms in Lagrangian (2) that results in a theory of the Chern–Simons type with only first order time derivatives. Then they introduce an additional potential $V(\vec{x}_1)$ describing an interaction of the first charge with an “impurity” centered at the origin and show that the matrix element of $V(\vec{x}_1)$ between dipole states contains the usual Moyal phase that is a signature of NCFT.

Notice that this result is obtained when the limit $m \rightarrow 0$ is taken directly in the Lagrangian. Let us now show that when one first solves this problem for a nonzero m and then takes the limit $m \rightarrow 0$ in the solution, additional exponential factors occur in the matrix element of $V(\vec{x}_1)$.

The Hamiltonian in model (1) is given by

$$H = \frac{\hat{\vec{p}}^2 + \hat{\vec{d}}^2}{4m} - \frac{eB}{m}\epsilon^{ab}\Delta^a\hat{p}^b + \left(\frac{e^2B^2}{m} + 2K\right)\vec{\Delta}^2, \quad (3)$$

where $\hat{\vec{p}}$ and $\hat{\vec{d}}$ are operators of the center of mass and relative momenta. Since the Hamiltonian is independent of the center of mass coordinates, the wave function can be represented in the form $\psi(\vec{X}, \vec{\Delta}) = e^{i\vec{p}\vec{X}}f(\vec{\Delta})$. Then, for $f(\vec{\Delta})$ we get the equation

$$\left(\frac{\hat{\vec{d}}^2}{4m} - \frac{eB}{m}\epsilon^{ab}\Delta^a\hat{p}^b + \left(\frac{e^2B^2}{m} + 2K\right)\vec{\Delta}^2\right)f(\vec{\Delta}) = (E - \frac{\vec{p}^2}{4m})f(\vec{\Delta}). \quad (4)$$

Changing the variables to

$$x = \Delta_x + \frac{eBp_y}{2(e^2B^2 + 2Km)},$$

$$y = \Delta_y - \frac{eBp_x}{2(e^2B^2 + 2Km)},$$

we arrive at the equation

$$\left(\frac{K\vec{p}^2}{2(e^2B^2 + 2Km)} - \frac{\partial_x^2 + \partial_y^2}{4m} + \left(\frac{e^2B^2}{m} + 2K \right) (x^2 + y^2) \right) f(x, y) = Ef(x, y). \quad (5)$$

Clearly, the first term here is the kinetic energy of the center of mass. Note that as $m \rightarrow 0$, it coincides with the eigenvalue of the Hamiltonian in Ref. [3] obtained from Lagrangian (1) with $m = 0$. All other terms that are present in Hamiltonian (5) [and reflecting the inner structure of composite states] are absent in the Hamiltonian of Ref. [3]. In that case, the only information about the inner structure of composites that is retained is given by the relation

$$\Delta^a = -\frac{\epsilon^{ab}p^b}{2eB}, \quad (6)$$

which expresses relative coordinates through the center of mass momentum.

Obviously, equation (5) admits an exact solution. Its spectrum contains an infinite number of composites (neutral bound states) with the energy eigenvalues

$$E_{\vec{p}, n, k} = \frac{K\vec{p}^2}{2r^2} + (n + k + 1)\frac{r}{m}, \quad (7)$$

where $r = \sqrt{e^2B^2 + 2Km}$ and n and k are positive integers or zero. Note that in the limit $K \rightarrow 0$ the Lagrangian (1) reduces to the Lagrangian of two noninteracting charged particles in a constant magnetic field (the Landau problem) and Eq.(7) correctly reproduces the Landau spectrum.

Thus, the model (1) describes an infinite number of neutral composites. The vector \vec{p} is their center of mass momentum and the last term in (7) reflects their nontrivial inner structure. Now, in the limit $m \rightarrow 0$, only the LLL states with $n = k = 0$ survive (all higher excitations decouple). The normalized LLL wave function with the center of mass momentum \vec{p} is given by

$$\langle \vec{X}, \vec{\Delta} | \vec{p} \rangle = \psi_{\vec{p}, 0, 0}(\vec{X}, \vec{\Delta}) = \left(\frac{r}{2\pi^3} \right)^{1/2} e^{i\vec{p}\vec{X}} e^{-r(\Delta_x + \frac{eBp_y}{2r^2})^2} e^{-r(\Delta_y - \frac{eBp_x}{2r^2})^2}. \quad (8)$$

The Gaussian exponential factors here reflect the inner structures of the composites. It is important that in the limit $m \rightarrow 0$, this wave function does *not* coincide with the wave function of Ref. [3] corresponding to the model with the Lagrangian (1) at $m = 0$: there are no Gaussian exponential factors in that case. In other words, while in the $m \rightarrow 0$ model, there are quantum fluctuations described by the Gaussian exponents, these fluctuations are completely suppressed in the model with $m \equiv 0$.

Thus we conclude that the quantum dynamics in the limit $m \rightarrow 0$ in the massive model does not coincide with that in the massless one. Recall that the same situation takes place in non-abelian gauge theories: the limit $m \rightarrow 0$ in a massive non-abelian model does not yield the dynamics of the massless one [11]. The origin of this phenomenon is the same in both cases. Because of constraints in the massless models, the number of physical degrees of freedom there is less than the number of degrees of freedom in the massive ones. In the present quantum mechanical model, these constraints are described by equation (6).

Of course, there is nothing wrong with the model (1) at $m = 0$. It is mathematically consistent. However, its dynamics is very different from that of a physical dipole in a strong magnetic field [by a physical dipole, we understand a dipole composed of two massive charged particle, including the case of an infinitesimally small mass $m \rightarrow 0$]. We also would like to point out that the present treatment of the dynamics in a strong magnetic field is equivalent to the formalism of the projection onto the LLL developed in Refs. [12, 13].

If following Ref. [3] we introduce an additional potential $V(\vec{x}_1)$ describing an interaction of the first charge with an impurity, the matrix element $\langle \vec{k} | V(\vec{x}_1) | \vec{p} \rangle$ will describe the scattering of composites on the impurity in the Born approximation. In order to evaluate $\langle \vec{k} | V(\vec{x}_1) | \vec{p} \rangle$, it is convenient to introduce the Fourier transform

$$V(\vec{x}_1) = \int \frac{d^2q}{(2\pi)^2} \tilde{V}(\vec{q}) e^{i\vec{q}\vec{x}_1}, \quad (9)$$

so that

$$\langle \vec{k} | V(\vec{x}_1) | \vec{p} \rangle = \int \frac{d^2q}{(2\pi)^2} \tilde{V}(\vec{q}) \langle \vec{k} | e^{i\vec{q}(\vec{X} + \vec{\Delta})} | \vec{p} \rangle. \quad (10)$$

Inserting now a complete set $\int d^2X d^2\Delta |\vec{X}, \vec{\Delta}\rangle \langle \vec{X}, \vec{\Delta}|$ and using Eq. (8), one can easily calculate the matrix element $\langle \vec{k} | e^{i\vec{q}(\vec{X}+\vec{\Delta})} | \vec{p} \rangle$. In the limit $m \rightarrow 0$, it is:

$$\langle \vec{k} | e^{i\vec{q}(\vec{X}+\vec{\Delta})} | \vec{p} \rangle = \delta^2(\vec{k} - \vec{q} - \vec{p}) e^{-\frac{\vec{q}^2}{4|eB|}} e^{-\frac{i}{2}\vec{q} \times \vec{k}} \quad (11)$$

with the cross product $\vec{q} \times \vec{k} \equiv \epsilon^{ab} q^a k^b / eB$. One can see that, in addition to the standard Moyal factor $e^{-\frac{i}{2}\vec{q} \times \vec{k}}$, this vertex contains also the exponentially damping term $e^{-\frac{\vec{q}^2}{4|eB|}}$. This term of course occurs due to the Gaussian factors in the wave function (8). It would be absent if we, as in Ref. [3], used the Lagrangian with $m = 0$ in Eq. (1).

The general character of this phenomenon suggests that additional exponential terms in interaction vertices should also occur in field theories in a strong magnetic field. This expectation will be confirmed in the next Section where it will be also shown that these theories determine a class of modified NCFT.

III. THE NJL MODEL IN A MAGNETIC FIELD AS A NCFT: THE EFFECTIVE ACTION

In this Section, we will consider the dynamics in the $(3 + 1)$ -dimensional NJL model in a strong magnetic field. Our aim is to show that this dynamics determines a consistent NCFT. As it will be shown in Section V, a similar situation takes place in an arbitrary dimension $D = d + 1$ with the space dimension $d \geq 2$.

The Lagrangian density of the NJL model with the $U_L(1) \times U_R(1)$ chiral symmetry reads

$$L = \frac{1}{2} [\bar{\psi}, (i\gamma^\mu D_\mu) \psi] + \frac{G}{2N} [(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\psi)^2], \quad (12)$$

where fermion fields carry an additional “color” index $i = 1, 2, \dots, N$ and the covariant derivative D_μ is

$$D_\mu = \partial_\mu - ieA_\mu^{ext}. \quad (13)$$

The external vector potential A_μ^{ext} describes a constant magnetic field B directed in the $+x^3$ direction. We will use two gauges in this paper: the symmetric gauge

$$A_\mu^{ext} = (0, \frac{Bx^2}{2}, -\frac{Bx^1}{2}, 0) \quad (14)$$

and the Landau gauge

$$A_\mu^{ext} = (0, Bx^2, 0, 0). \quad (15)$$

We will consider the dynamics of neutral bound states (“dipoles”) in this model with large N , when the $1/N$ expansion is justified. In this case, the model becomes essentially soluble. The central dynamical phenomenon in the model is the phenomenon of the magnetic catalysis: a constant magnetic field is a strong catalyst of dynamical chiral symmetry breaking, leading to the generation of a fermion dynamical mass even at the weakest attractive interaction between fermions [14, 15]. The essence of this effect is the dimensional reduction in the dynamics of fermion pairing in a strong magnetic field, when the LLL dynamics dominates. In the original papers [14, 15], the dynamics in the dimensions $D = 2 + 1$ and $D = 3 + 1$ were considered. That analysis was extended to the case of a general space dimension $d \geq 2$ in Ref. [16] [for earlier consideration of dynamical symmetry breaking in a magnetic field, see Refs. [17, 18]].

It is well known that in the model (12) with large N , the relevant neutral degrees of freedom are connected with the composite fields $\sigma \sim \bar{\psi}\psi$ and $\pi \sim \bar{\psi}i\gamma_5\psi$. The action for them has the following form [15]:

$$\Gamma(\sigma, \pi) = -iTr \text{Ln} (i\gamma^\mu D_\mu - (\sigma + i\gamma^5\pi)) - \frac{N}{2G} \int d^4x (\sigma^2 + \pi^2). \quad (16)$$

The gap equation for $\langle 0 | \sigma | 0 \rangle = m$, where m is the fermion dynamical mass, is

$$\frac{\partial}{\partial \sigma} V(\sigma, \pi) |_{\sigma=m, \pi=0} = 0, \quad (17)$$

where $V(\sigma, \pi)$ is the potential connected with action (16). According to [15], in a magnetic field, this equation has a non-zero solution for the mass m for an arbitrary positive G , i.e., the critical coupling constant equals zero in this problem.

The LLL dominance takes place in the weak coupling regime, with the dimensionless coupling constant $g \equiv G\Lambda^2/4\pi^2 \ll 1$. In this case, the dynamical mass m is [15]

$$m^2 = \Lambda^2 e^{-\frac{4\pi^2}{|eB|G}} = \Lambda^2 e^{-\frac{\Lambda^2}{|eB|g}}, \quad (18)$$

where Λ is an ultraviolet cutoff connected with longitudinal momenta $k_{\parallel} = (k^0, k^3)$ (we assume that $\Lambda^2 \gg |eB|$).¹ Notice that Eq. (18) implies the following hierarchy of scales: $\frac{|eB|}{m^2} \gg \frac{\Lambda^2}{|eB|}$. It will be shown in Section IV that a meaningful continuum limit $\Lambda^2 = C|eB| \rightarrow \infty$, with $C \gg 1$ and m being fixed, exists in this model.

It is straightforward to calculate the interaction vertices for the $\tilde{\sigma} = \sigma - m$ and π fields that follows from action (16). For example, the 3-point vertex $\Gamma_{\tilde{\sigma}\pi\pi}$ is given by

$$\Gamma_{\tilde{\sigma}\pi\pi} = \int d^4x d^4y d^4z \text{tr}[S(x, y) \gamma^5 \pi(y) S(y, z) \gamma^5 \pi(z) S(z, x) \tilde{\sigma}(x)]. \quad (19)$$

Here the LLL fermion propagator $S(x, y)$ is equal to [15]

$$S(x, y) = e^{\frac{i}{2}(x-y)^\mu A_\mu^{ext}(x+y)} \tilde{S}(x-y), \quad (20)$$

where the Fourier transform of the translationally invariant part \tilde{S} is

$$\tilde{S}(k) = i e^{-\frac{k^2}{|eB|}} \frac{k^0 \gamma^0 - k^3 \gamma^3 + m}{k_0^2 - k_3^2 - m^2} (1 - i \gamma^1 \gamma^2 \text{sign}(eB)), \quad (21)$$

[for convenience, the δ -symbol with color indices in the propagator is omitted]. The first factor in (20) is the Schwinger phase factor [19]. It breaks the translation invariance even in the case of a constant magnetic field, although in this case there is a group of magnetic translations whose generators, unlike usual momenta, do not commute.

The Schwinger phase $\phi = \frac{i}{2}(x-y)^\mu A_\mu^{ext}(x+y)$ is equal to

$$\phi_{sym} = \frac{ieB}{2} \epsilon^{ab} x^a y^b, \quad a, b = 1, 2 \quad (22)$$

in the symmetric gauge (14), and it is

$$\phi_{Landau} = \phi_{sym} + \frac{ieB}{2} (x^1 x^2 - y^1 y^2) \quad (23)$$

in the Landau gauge (15). One can easily check that the total phase along the closed fermion loop in (19) is gauge invariant, i.e., independent of a gauge.

We will show that, in the regime with the LLL dominance, the effective action (16) leads to a NCFT with noncommutative space transverse coordinates x^a :

$$[\hat{x}^a, \hat{x}^b] = i \frac{1}{eB} \epsilon^{ab} \equiv i \theta^{ab}. \quad (24)$$

It is the Schwinger phase that is responsible for this noncommutativity. Indeed, the commutator $[\hat{x}^a, \hat{x}^b]$ is of course antisymmetric and the only place where an antisymmetric tensor occurs in 3-point vertex (19) is the Schwinger phase (as will be shown below, a similar situation takes place also for higher vertices).

We begin our analysis with the observation that the LLL fermion propagator (20) factorizes into two parts, the part depending on the transverse coordinates $x_{\perp} = (x^1, x^2)$ and that depending on the longitudinal coordinates $x_{\parallel} = (x^0, x^3)$:

$$S(x, y) = P(x_{\perp}, y_{\perp}) S_{\parallel}(x_{\parallel} - y_{\parallel}). \quad (25)$$

Indeed, taking into account expressions (20), (21), and (22), we get in the symmetric gauge:

$$P(x_{\perp}, y_{\perp}) = \frac{|eB|}{2\pi} e^{\frac{ieB}{2} \epsilon^{ab} x^a y^b} e^{-\frac{|eB|}{4} (\vec{x}_{\perp} - \vec{y}_{\perp})^2} \quad (26)$$

¹ As will become clear below, there are no divergences connected with transverse momenta $\vec{k}_{\perp} = (k^1, k^2)$ in the regime with the LLL dominance, and therefore the ‘‘longitudinal’’ cutoff removes all divergences in this model.

and

$$S_{\parallel}(x_{\parallel} - y_{\parallel}) = \int \frac{d^2 k_{\parallel}}{(2\pi)^2} e^{ik_{\parallel}(x_{\parallel} - y_{\parallel})} \frac{i}{k_{\parallel}\gamma^{\parallel} - m} \frac{1 - i\gamma^1\gamma^2\text{sign}(eB)}{2} \quad (27)$$

[henceforth, for concreteness, we will use the symmetric gauge]. The longitudinal part $S_{\parallel}(x_{\parallel} - y_{\parallel})$ is nothing else but a fermion propagator in 1+1 dimensions. In particular, the matrix $(1 - i\gamma^1\gamma^2\text{sign}(eB))/2$ is the projector on the fermion (antifermion) states with the spin polarized along (opposite to) the magnetic field, and therefore it projects on two states of the four ones, as should be in 1+1 dimensions. As to the operator $P(x_{\perp}, y_{\perp})$, it is easy to check that it satisfies the relation

$$\int d^2 y^{\perp} P(x^{\perp}, y^{\perp}) P(y^{\perp}, z^{\perp}) = P(x^{\perp}, z^{\perp}) \quad (28)$$

and therefore is a projection operator. Since $S(x, y)$ is a LLL propagator, it is clear that $P(x_{\perp}, y_{\perp})$ is a projection operator on the LLL states.

The factorization of the LLL propagator leads to a simple structure of interaction vertices for π and $\tilde{\sigma}$ fields. For example, as it is clear from expression (19) for the 3-point vertex, a substitution of the Fourier transforms for the fields makes the integration over the longitudinal and transverse coordinates completely independent. And since S_{\parallel} is a $(1 + 1)$ -dimensional propagator, the integration over x_{\parallel} coordinates yields a fermion loop in the $(1+1)$ -dimensional Minkowski space. It is obvious that the same is true also for higher order interaction vertices arising from action (16). Therefore the dependence of interaction vertices on longitudinal k_{\parallel} momenta is standard and, for clarity of the presentation, we will first consider the case with all external longitudinal momenta entering the fermion loop to be zero. This of course corresponds to the choice of π and $\tilde{\sigma}$ fields independent of longitudinal coordinates x_{\parallel} . The general case, with the fields depending on both transverse and longitudinal coordinates, will be considered in the end of this Section.

Now, substituting the Fourier transforms of the fields $\pi(x_{\perp})$ and $\tilde{\sigma}(x_{\perp})$ into Eq. (19) and using Eqs. (25), (26), and (27), we find the following expression for the 3-point interaction vertex $\Gamma_{\tilde{\sigma}\pi\pi}$ in the momentum space:

$$\begin{aligned} \Gamma_{\tilde{\sigma}\pi\pi} = & -\frac{N|eB|}{m} \int d^2 x_{\parallel} \int \frac{d^2 k_1 d^2 k_2 d^2 k_3}{(2\pi)^6} \pi(k_1) \pi(k_2) \tilde{\sigma}(k_3) \delta^2(k_1 + k_2 + k_3) \\ & \times e^{-\frac{k_1^2 + k_2^2 + k_3^2}{4|eB|}} \exp\left[-\frac{i}{2}(k_1 \times k_2 + k_1 \times k_3 + k_2 \times k_3)\right], \end{aligned} \quad (29)$$

where $k_i \times k_j = k_i^a \theta^{ab} k_j^b \equiv k_i \theta k_j$, $\theta^{ab} = \frac{1}{eB} \epsilon^{ab}$ (here, for convenience, we omitted the subscript \perp for the transverse coordinates). Notice that because of the exponentially damping factors, there are no ultraviolet divergences in this expression.

According to [1], an n -point vertex in a noncommutative theory in momentum space has the following structure:

$$\int \frac{d^D k_1}{(2\pi)^D} \dots \frac{d^D k_n}{(2\pi)^D} \phi(k_1) \dots \phi(k_n) \delta^D\left(\sum_i k_i\right) e^{-\frac{i}{2} \sum_{i < j} k_i \times k_j}, \quad (30)$$

where here ϕ denotes a generic field and the exponent $e^{-\frac{i}{2} \sum_{i < j} k_i \times k_j} \equiv e^{-\frac{i}{2} \sum_{i < j} k_i \theta k_j}$ is the Moyal exponent factor. Comparing expressions (29) and (30), we see that apart from the factor $e^{-\frac{k_1^2 + k_2^2 + k_3^2}{4|eB|}}$, the vertex $\Gamma_{\tilde{\sigma}\pi\pi}$ coincides with the standard 3-point interaction vertex in a noncommutative theory with the commutator $[x^a, x^b] = i\theta^{ab} = \frac{i}{eB} \epsilon^{ab}$. In order to take properly into account this additional factor in the vertex, it will be convenient to introduce new, “smeared”, fields:

$$\Pi(x) = e^{\frac{\nabla_{\perp}^2}{4|eB|}} \pi(x), \quad \Sigma(x) = e^{\frac{\nabla_{\perp}^2}{4|eB|}} \sigma(x), \quad (31)$$

where ∇_{\perp}^2 is the transverse Laplacian. Then, in terms of these fields, the vertex can be rewritten in the standard form with the Moyal exponent factor:

$$\Gamma_{\tilde{\Sigma}\Pi\Pi} = -\frac{N|eB|}{m} \int d^2 x_{\parallel} \int \frac{d^2 k_1 d^2 k_2 d^2 k_3}{(2\pi)^6} \Pi(k_1) \Pi(k_2) \tilde{\Sigma}(k_3) \delta^2\left(\sum_i k_i\right) \exp\left[-\frac{i}{2} \sum_{i < j} k_i \times k_j\right]. \quad (32)$$

One can similarly analyze the 4-point interaction vertex $\Gamma_{4\Pi}$. We get:

$$\Gamma_{4\Pi} = -\frac{N|eB|}{4m^2} \int d^2x_{||} \int \frac{d^2k_1 d^2k_2 d^2k_3 d^2k_4}{(2\pi)^8} \Pi(k_1) \Pi(k_2) \Pi(k_3) \Pi(k_4) \delta^2\left(\sum_i k_i\right) \exp\left[-\frac{i}{2} \sum_{i<j} k_i \times k_j\right]. \quad (33)$$

The occurrence of the smeared fields in the vertices reflects an inner structure (dynamical form-factors) of π and σ composites on the lowest Landau level, which is similar to that of a dipole in the quantum mechanical problem considered in Section II.

As is well known, the cross product in the momentum space corresponds to a star product in the coordinate space [1]:

$$(\Phi * \Phi)(x) = e^{\frac{i}{2}\theta^{ab} \frac{\partial}{\partial y^a} \frac{\partial}{\partial z^b}} \Phi(y) \Phi(z)|_{y=z=x}, \quad (34)$$

where here Φ represents the smeared fields Π and Σ . By using the star product, one can rewrite the vertices $\Gamma_{\Sigma\Pi\Pi}$ and $\Gamma_{4\Pi}$ in the following simple form in the coordinate space:

$$\begin{aligned} \Gamma_{\Sigma\Pi\Pi} &= -\frac{N|eB|}{4\pi^2 m} \int d^2x_{||} d^2x_{\perp} \tilde{\Sigma} * \Pi * \Pi, \\ \Gamma_{4\Pi} &= -\frac{N|eB|}{16\pi^2 m^2} \int d^2x_{||} d^2x_{\perp} \Pi * \Pi * \Pi * \Pi. \end{aligned} \quad (35)$$

As to expressing the vertices in NCFT in the space with noncommutative coordinates \hat{x}^a , one should use the Weyl symbol of a field $\hat{\Phi}$ there [1]:

$$\hat{\Phi}(\hat{x}) \equiv \hat{W}[\Phi] = \int d^D x \Phi(x) \hat{\Delta}(x), \quad \hat{\Delta}(x) \equiv \int \frac{d^D k}{(2\pi)^D} e^{ik_a \hat{x}^a} e^{-ik_a x^a}. \quad (36)$$

The most important property of the Weyl symbol is that the product of the Weyl symbols of two functions is equal to the Weyl symbol of their star product:

$$\hat{W}[\Phi_1] \hat{W}[\Phi_2] = \hat{W}[\Phi_1 * \Phi_2]. \quad (37)$$

In our case, the Weyl symbol $\hat{\Phi}$ represents $\hat{\Pi}$ and $\hat{\Sigma}$. Note that the relation between the Weyl symbols of smeared and non-smeared fields is:

$$\hat{\Phi}(\hat{x}) = e^{\frac{\hat{\nabla}_{\perp}^2}{4|eB|}} \hat{\phi}(\hat{x}), \quad (38)$$

where the operator $\hat{\nabla}_{\perp}^2$ in the noncommutative space acts as

$$\hat{\nabla}_{\perp}^2 \hat{\phi}(\hat{x}) = -(eB)^2 \sum_{a=1}^2 [\hat{x}^a, [\hat{x}^a, \hat{\phi}(\hat{x})]] \quad (39)$$

[the latter relation follows from the definition of the derivative in NCFT, $\hat{\nabla}_{\perp a} \hat{\phi}(\hat{x}) = -i[(\theta^{-1})_{ab} \hat{x}^b, \hat{\phi}(\hat{x})]$ [1]].

In terms of $\hat{\Phi}$, the 3- and 4-point vertices take the following form in NCFT:

$$\begin{aligned} \Gamma_{\Sigma\Pi\Pi} &= -\frac{N|eB|}{4\pi^2 m} \int d^2x_{||} \mathbf{Tr} \hat{\Sigma} \hat{\Pi}^2, \\ \Gamma_{4\Pi} &= -\frac{N|eB|}{16\pi^2 m^2} \int d^2x_{||} \mathbf{Tr} \hat{\Pi}^4, \end{aligned} \quad (40)$$

where the operation \mathbf{Tr} is defined as in [1]. As is shown in Appendix A, all interaction vertices $\Gamma_{n\Phi}$ ($n \geq 3$) arising from action (16) have a similar structure.

There exists another, more convenient for practical calculations, representation of interaction vertices in which the vertices are expressed through the initial, “non-smeared”, fields π and $\tilde{\sigma}$. The point is that, due to the presence

of the δ -function $\delta^2(\sum_i k_i)$, the exponent factors $e^{-\frac{\sum_{i=1}^n \vec{k}_i^2}{4|eB|}} e^{-\frac{i}{2} \sum_{i < j} k_i \times k_j}$ in an n -point vertex can be rewritten as $e^{-\frac{i}{2} \sum_{i < j} k_i \times_M k_j}$, where $k_i \times_M k_j$ is a new cross product. It is defined as

$$k_i \times_M k_j = k_i \Omega k_j \quad (41)$$

with the matrix Ω being

$$\Omega^{ab} = \frac{1}{|eB|} \begin{pmatrix} i & \text{sign}(eB) \\ -\text{sign}(eB) & i \end{pmatrix}. \quad (42)$$

We will call $k_i \times_M k_j$ an M (magnetic)-cross product. Notice that like the matrix θ^{ab} , defining the cross product, the new matrix Ω^{ab} , defining the M -cross product, is anti-hermitian.

By using the M -cross product, we get the following simple structure for an n -point vertex in the momentum space:

$$\int d^2 x_{||} \frac{d^2 k_1}{(2\pi)^2} \dots \frac{d^2 k_n}{(2\pi)^2} \phi(k_1) \dots \phi(k_n) \delta^2(\sum_i k_i) e^{-\frac{i}{2} \sum_{i < j} k_i \times_M k_j} \quad (43)$$

(compare with expression (30)). Here the field ϕ represents initial fields π and $\tilde{\sigma}$.

In the coordinate space, the M -cross product becomes an M -star product:

$$(\phi *_M \phi)(x) = e^{\frac{i}{2} \Omega^{ab} \frac{\partial}{\partial y^a} \frac{\partial}{\partial z^b}} \phi(y) \phi(z) |_{y=z=x}. \quad (44)$$

(compare with equation (34)). By using the M -star product, one can express n -point vertices through the initial π and $\tilde{\sigma}$ fields in the coordinate space. For example, the vertices $\Gamma_{\tilde{\sigma}\pi\pi}$ and $\Gamma_{4\pi}$ become:

$$\begin{aligned} \Gamma_{\tilde{\sigma}\pi\pi} &= -\frac{N|eB|}{4\pi^2 m} \int d^2 x_{||} d^2 x_{\perp} \tilde{\sigma} *_M \pi *_M \pi, \\ \Gamma_{4\pi} &= -\frac{N|eB|}{16\pi^2 m^2} \int d^2 x_{||} d^2 x_{\perp} \pi *_M \pi *_M \pi *_M \pi. \end{aligned} \quad (45)$$

In fact, by using the M -star product, the whole effective action (16) can be written in a compact and explicit form for the case of fields independent of longitudinal coordinates $x_{||}$. First, note that for constant fields, the M -star product in $\Gamma_{n\phi}$ vertices (45) is reduced to the usual product and the vertices come from the effective potential in that case. Then, this implies that, up to the measure $-\int d^4 x$, the whole effective action for fields depending on transverse coordinates coincides with the effective potential in which the usual product is replaced by the M -star product in the part coming from the $Tr \ln$ term in (16). As to the last term $\frac{N}{2G} \int d^4 x (\sigma^2 + \pi^2)$ there, it should stay as it is. This is because unlike the star product, the M -star product and the usual one lead to different quadratic terms in the action. Now, by using expression (20) for the fermion propagator, we easily find the effective potential:

$$V(\sigma, \pi) = \frac{N|eB|}{8\pi^2} [\sigma^2 + \pi^2] \left[\ln\left(\frac{\sigma^2 + \pi^2}{\Lambda^2}\right) - 1 \right] + \frac{N}{2G} (\sigma^2 + \pi^2) + O\left(\frac{\sigma^2 + \pi^2}{\Lambda^2}\right). \quad (46)$$

Then, the effective action reads:

$$\Gamma(\sigma, \pi) = -\frac{N|eB|}{8\pi^2} \int d^4 x \left([\sigma^2 + \pi^2] \left[\ln\left(\frac{\sigma^2 + \pi^2}{\Lambda^2}\right) - 1 \right] \right)_{*_M} - \frac{N}{2G} \int d^4 x (\sigma^2 + \pi^2). \quad (47)$$

This expression is very convenient for calculating the n -point vertices $\Gamma_{n\phi}$. In Appendix B, it is shown that the M -star product also naturally appears in the formalism of the projected density operators developed in Ref. [21] for the description of the quantum Hall effect.

While the M -star product is useful for practical calculations, its connection with the multiplication operation in a noncommutative coordinate space is not direct. This is in contrast with the star product for which relation (37) takes place. Therefore it will be useful to rewrite action (47) through the star product. It can be done by using the smeared fields Σ and Π . The result is:

$$\Gamma = -\frac{N|eB|}{8\pi^2} \int d^4 x \left([\Sigma^2 + \Pi^2] \left[\ln\left(\frac{\Sigma^2 + \Pi^2}{\Lambda^2}\right) - 1 \right] + \frac{4\pi^2}{G|eB|} (\sigma^2 + \pi^2) \right)_{*}, \quad (48)$$

where we used the fact that the star product and the usual one lead to the same quadratic terms in the action. Notice that the fields σ and π are connected with the smeared fields through the non-local relation (31). This relation implies that exponentially damping form-factors are built in the propagators of the smeared fields. As a result, their propagators decrease rapidly, as $\exp(-k_\perp^2/2|eB|)$, with $k_\perp^2 \rightarrow \infty$. As will be shown in the next Section, this property is in particular responsible for removing the UV/IR mixing in the model.

By using relation (37), it is straightforward to rewrite the action in the NCFT through Weyl symbols:

$$\Gamma = -\frac{N|eB|}{8\pi^2} \int d^2x_{||} \text{Tr} \left([\hat{\Sigma}^2 + \hat{\Pi}^2] \left[\ln\left(\frac{\hat{\Sigma}^2 + \hat{\Pi}^2}{\Lambda^2}\right) - 1 \right] + \frac{4\pi^2}{G|eB|} (\hat{\sigma}^2 + \hat{\pi}^2) \right). \quad (49)$$

The Weyl symbols $\hat{\sigma}$ and $\hat{\pi}$ are connected with the Weyl symbols $\hat{\Sigma}$ and $\hat{\Pi}$ through relation (38).

Let us now generalize expressions (47), (48), and (49) to the case when fields π and σ depend on both transverse and longitudinal coordinates. First, as it follows from the expansion of the first term in action (16) in a series in π and $\tilde{\sigma}$, the general n -point vertex is given by

$$\begin{aligned} \Gamma_{n\phi} = & -\frac{(-i)^{n+1}N}{n} \int d^2x_1^\perp \dots d^2x_n^\perp d^2x_1^\parallel \dots d^2x_n^\parallel P(x_1^\perp, x_2^\perp) \dots P(x_n^\perp, x_1^\perp) \\ & \times \text{tr} [S_{||}(x_1 - x_2)(\tilde{\sigma}(x_2) + i\gamma^5\pi(x_2)) \dots S_{||}(x_n - x_1)(\tilde{\sigma}(x_1) + i\gamma^5\pi(x_1))] \end{aligned} \quad (50)$$

[notice that here the longitudinal part of the fermion propagator $S_{||}(x)$ does not contain color indices]. As was shown above and in Appendix A, the transverse part of vertex (50) can be expressed through the M -star product. Therefore the n -point vertex is

$$\begin{aligned} \Gamma_{n\phi} = & -\frac{(-i)^{n+1}N|eB|}{2\pi n} \int d^2x^\perp d^2x_1^\parallel \dots d^2x_n^\parallel \\ & \times \text{tr} \left[S_{||}(x_1 - x_2)(\tilde{\sigma}(x_1^\perp, x_2^\parallel) + i\gamma^5\pi(x_1^\perp, x_2^\parallel)) \dots S_{||}(x_n - x_1)(\tilde{\sigma}(x_n^\perp, x_1^\parallel) + i\gamma^5\pi(x_n^\perp, x_1^\parallel)) \right]_{*M}. \end{aligned} \quad (51)$$

This relation implies that the full effective action can be written through the M -star product as:

$$\Gamma(\sigma, \pi) = -\frac{iN|eB|}{2\pi} \int d^2x_\perp \text{Tr}_{||} \left[\mathcal{P} \text{Ln} \left(i\gamma^\parallel \partial_\parallel - (\sigma + i\gamma^5\pi) \right) \right]_{*M} - \frac{N}{2G} \int d^4x (\sigma^2 + \pi^2), \quad (52)$$

where the projector \mathcal{P} is

$$\mathcal{P} \equiv \frac{1 - i\gamma^1\gamma^2\text{sign}(eB)}{2} \quad (53)$$

(compare with action (47)). Here the trace $\text{Tr}_{||}$, related to the longitudinal subspace, is taken in the functional sense.

As to the form of the effective action written through the star product and its form in the noncommutative coordinate space, they are:

$$\Gamma = \frac{N|eB|}{2\pi} \int d^2x_\perp \left[-i\text{Tr}_{||} \left[\mathcal{P} \text{Ln} \left(i\gamma^\parallel \partial_\parallel - (\Sigma + i\gamma^5\Pi) \right) \right] - \frac{\pi}{G|eB|} \int d^2x_{||} (\sigma^2 + \pi^2) \right]_* \quad (54)$$

and

$$\Gamma = \frac{N|eB|}{2\pi} \text{Tr} \left[-i\text{Tr}_{||} \left[\mathcal{P} \text{Ln} \left(i\gamma^\parallel \partial_\parallel - (\hat{\Sigma} + i\gamma^5\hat{\Pi}) \right) \right] - \frac{\pi}{G|eB|} \int d^2x_{||} (\hat{\sigma}^2 + \hat{\pi}^2) \right] \quad (55)$$

(compare with Eqs. (48) and (49), respectively).

This concludes the derivation of the action of the noncommutative field theory corresponding to the NJL model in a strong magnetic field. In the next Section, we will consider the dynamics in this model in more detail.

IV. THE NJL MODEL IN A MAGNETIC FIELD AS A NCFT: THE DYNAMICS

In the regime with the LLL dominance, the dynamics of neutral composites is described by quite sophisticated NCFT (55). In this Section, we will show that in this model *i*) there exists a well defined commutative limit $|eB| \rightarrow \infty$ when $[\hat{x}^a, \hat{x}^b] = 0$; *ii*) the universality class of the low energy dynamics, with $k_\perp \ll \sqrt{|eB|}$, is intimately connected with the dynamics in the (1+1)-dimensional Gross-Neveu (GN) model [20]; and *iii*) there is no UV/IR mixing.

The key point in the derivation of action (55) was the fact that the LLL fermion propagator (20) factorizes into two parts (see Eq. (25)) and that its transverse part $P(x_\perp, y_\perp)$ is a projection operator on the LLL states. It is quite remarkable that it exactly coincides with the projection operator on the LLL states in nonrelativistic dynamics introduced for the description of the quantum Hall effect in Refs. [12, 22].² Therefore the transverse dynamics in this problem is universal and peculiarities of the relativistic dynamics reflect themselves only in the $(1+1)$ -dimensional longitudinal space.

In order to study the low energy dynamics with $k_\perp \ll \sqrt{|eB|}$, it will be instructive to consider, as in Ref. [23], the following continuum limit: $\Lambda^2 = C|eB| \rightarrow \infty$, with $C \gg 1$ and m being fixed. Let us consider n -point vertex (50) in this limit. Since the projection operator $P(x_i^\perp, x_{i+1}^\perp)$ is

$$P(x_i^\perp, x_{i+1}^\perp) = \frac{|eB|}{2\pi} e^{\frac{i e B}{2} \epsilon^{ab} x_i^a x_{i+1}^b} e^{-\frac{|eB|}{4} (\vec{x}_i^\perp - \vec{x}_{i+1}^\perp)^2} \quad (56)$$

(see Eq. (26)), the point with coordinates $x_i^\perp = x_{i+1}^\perp$, $i = 1, \dots, n-1$, is both a saddle and stationary point in the multiple integral (50) in the limit $|eB| \rightarrow \infty$. Therefore, in order to get the leading term of the asymptotic expansion of that integral, one can put $x_i^\perp = x_n^\perp \equiv x^\perp$ in the arguments of all the fields $\tilde{\sigma}(x_i^\perp) + i\gamma^5 \pi(x_i^\perp)$ there. Then, by using relation (28) and the equality $P(x^\perp, x^\perp) = |eB|/2\pi$, we easily integrate over transverse coordinates in (50) and obtain the following asymptotic expression:

$$\begin{aligned} \Gamma_{n\phi}^{(as)} = & -\frac{(-i)^{n+1}}{n} \frac{N|eB|}{2\pi} \int d^2 x^\perp d^2 x_1^\parallel \dots d^2 x_n^\parallel \\ & \times \text{tr} \left[S_\parallel(x_1 - x_2) [\tilde{\sigma}(x^\perp, x_2^\parallel) + i\gamma^5 \pi(x^\perp, x_2^\parallel)] \dots S_\parallel(x_n - x_1) [\tilde{\sigma}(x^\perp, x_1^\parallel) + i\gamma^5 \pi(x^\perp, x_1^\parallel)] \right]. \end{aligned} \quad (57)$$

This equation implies that as $|eB| \rightarrow \infty$, the leading asymptotic term in the action is:

$$\Gamma^{(as)}(\sigma, \pi) = \frac{|eB|}{2\pi} \int d^2 x^\perp \left[-iN \text{Tr} r_\parallel \left[\mathcal{P} \text{Ln} \left(i\gamma^\parallel \partial_\parallel - (\sigma + i\gamma^5 \pi) \right) \right] - \frac{N\pi}{G|eB|} \int d^2 x_\parallel (\sigma^2 + \pi^2) \right]. \quad (58)$$

This action corresponds to a *commutative* field theory, as should be in the limit $|eB| \rightarrow \infty$ [indeed, the commutator $[\hat{x}^a, \hat{x}^b] = i\frac{1}{eB} \epsilon^{ab}$ goes to zero as $|eB| \rightarrow \infty$]. Also, since there is no hopping term for the transverse coordinates x^\perp in this action, they just play the role of a label of the fields.

Let us now compare this action with the action of the (1+1)-dimensional GN model [20]:

$$\Gamma_{GN}(\sigma, \pi) = -iN \text{Tr} \text{Ln} (i\gamma^\mu \partial_\mu - (\sigma + i\gamma^5 \pi)) - \frac{N}{2\tilde{G}} \int d^2 x (\sigma^2 + \pi^2), \quad \mu = 0, 1, \quad (59)$$

where \tilde{G} is a dimensionless coupling constant. One can see that, up to the factor $|eB|/2\pi \int d^2 x_\perp$, these two actions coincide if the constant G in (58) is identified with $2\pi\tilde{G}/|eB|$. In particular, with this identification, expression (18) for the dynamical mass coincides with the expression for m in the GN model, $m^2 = \Lambda^2 e^{-\frac{2\pi}{\tilde{G}}}$. Also, using Eq. (18), one can express the coupling constant G in the effective potential (46) through the dynamical mass m and cutoff Λ . Then, up to $O((\sigma^2 + \pi^2)/\Lambda^2)$ terms, we get the expression independent of the cutoff:

$$V(\sigma, \pi) = \frac{N|eB|}{8\pi^2} [\sigma^2 + \pi^2] \left[\ln \left(\frac{\sigma^2 + \pi^2}{m^2} \right) - 1 \right]. \quad (60)$$

This renormalized form of the potential coincides with the GN potential.

² This point is intimately connected with that the wave functions of the LLL states in the Landau problem are independent of the fermion mass m (see equation (8) for $K = 0$).

As to the factor $|eB|/2\pi \int d^2x_\perp$, its meaning is very simple. Since density of the LLL states is equal to $|eB|/2\pi$, this factor yields the number of the Landau states on the transverse plane. In other words, as $|eB| \rightarrow \infty$, the model is reduced to a continuum set of independent (1+1)-dimensional GN models, labeled by the coordinates in the plane perpendicular to the magnetic field. The conjecture about such a structure of the NJL model in the limit $|eB| \rightarrow \infty$ was made in Ref. [23] and was based on a study of the effective potential and the kinetic (two derivative) term in the model. The present approach allows to derive the whole action and thus to prove the conjecture.

The existence of the physically meaningful limit $|eB| \rightarrow \infty$ is quite noticeable. It confirms that the model with the LLL dominance is self-consistent. In order to understand its dynamics better, it is instructive to look at the dispersion relations for σ and π excitations with momenta $k_\perp \ll \sqrt{|eB|}$ [15]:

$$E_\pi \simeq \left[\frac{m^2}{|eB|} \ln \left(\frac{|eB|}{\pi m^2} \right) \vec{k}_\perp^2 + k_3^2 \right]^{1/2},$$

$$E_\sigma \simeq \left[12 m^2 + \frac{3m^2}{|eB|} \ln \left(\frac{|eB|}{\pi m^2} \right) \vec{k}_\perp^2 + k_3^2 \right]^{1/2}. \quad (61)$$

We find from these relations that the transverse velocity $|\vec{v}_\perp| = |\partial E_{\pi,\sigma} / \partial \vec{k}_\perp|$ of both π and σ goes rapidly (as $O(m^2/|eB|)$) to zero as $|eB| \rightarrow \infty$. In other words, there is no hopping between different transverse points in this limit. For a strong but finite magnetic field, the transverse velocity is, although nonzero, very small. In this case, the π and σ composites have a string-like shape: while their transverse size is of the order of the magnetic length $l = 1/\sqrt{|eB|}$, the longitudinal size is of order $1/m$, and $l \ll 1/m$.

The important point is that besides being a low energy theory of the initial NJL model in a magnetic field, this truncated [based on the LLL dynamics] model is self-contained. In particular, in this model one can consider arbitrary large values for transverse momenta k_\perp , although in this case its dynamics is very different from that of the initial NJL model. In fact, by using the expression for the pion propagator (66) written below, it is not difficult to check that for $k_\perp \gg \sqrt{|eB|}$ the dispersion relation for π excitations takes the following form:

$$E_\pi \simeq \left[4m^2 \left(1 - \frac{\pi^2 e^{-\frac{\vec{k}_\perp^2}{|eB|}}}{\ln^2 \frac{|eB|}{m^2}} \right) + k_3^2 \right]^{1/2}. \quad (62)$$

In this regime, the transverse velocity $|\vec{v}_\perp|$ is extremely small, $|\vec{v}_\perp| \sim \frac{m|\vec{k}_\perp|}{|eB|} e^{-\vec{k}_\perp^2/|eB|}$, and a π excitation is a loosely bound state moving along the x_3 direction. Its mass is close to the $2m$ threshold.

Thus we conclude that the NJL model in a strong magnetic field yields an example of a consistent NCFT with quite nontrivial dynamics. The point that exponentially damping factors occur either in vertices (for the fields σ and π) or in propagators (for the smeared fields) plays a crucial role in its consistency. Let us now show that these factors are in particular responsible for removing a UV/IR mixing, the phenomenon that plagues conventional nonsupersymmetric NCFT [10].

The simplest example of the UV/IR mixing is given by a one-loop contribution in a propagator in the noncommutative ϕ^4 model with the action

$$S = \int d^4x \left(\frac{1}{2} (\partial_\mu \phi)^2 - \frac{m^2 \phi^2}{2} - \frac{g^2}{4!} \phi * \phi * \phi * \phi \right). \quad (63)$$

There are planar and nonplanar one-loop contributions in the propagator of ϕ in this model [10]:

$$\Gamma_{nc}^{(2)} = \Gamma_{pl}^{(2)} + \Gamma_{npl}^{(2)} = \frac{g^2}{3(2\pi)^4} \int \frac{d^4k}{k^2 + m^2} + \frac{g^2}{6(2\pi)^4} \int \frac{d^4k}{k^2 + m^2} e^{ik \times p}. \quad (64)$$

The nonplanar contribution is specific for a noncommutative theory and is responsible for the UV/IR mixing. Indeed, the nonplanar contribution is equal to

$$\Gamma_{npl}^{(2)} = \frac{g^2}{96\pi^2} (\Lambda_{eff}^2 - m^2 \ln(\frac{\Lambda_{eff}^2}{m^2}) + O(1)), \quad (65)$$

where

$$\Lambda_{eff}^2 = \frac{1}{1/\Lambda^2 - p^i \theta_{ij}^2 p^j}$$

with Λ being cutoff. It is clear that if the external momentum $p \rightarrow 0$, the nonplanar contribution (65) diverges quadratically. On the other hand, for a nonzero p , it is finite due to the Moyal phase factor $e^{ik \times p}$ in the second term in expression (64) (which oscillates rapidly at large k). Thus, although the Moyal factor regularizes the UV divergence, it leads to an IR divergence of the integral, i.e., to the UV/IR mixing.

Let us now show how the exponentially damping factors in vertices (for the fields σ and π) or in propagators (for the smeared fields Σ and Π) remove the UV/IR mixing. We will first consider the description using the fields σ and π . As an example, we will consider the one-loop correction in the π propagator generated by the four-point interaction vertex $\Gamma_{4\pi}$. First, from action (16), we get this propagator in tree approximation. In Euclidean space it is:

$$D_{\pi}^{(tree)}(p) \simeq \frac{4\pi^2}{N|eB|[(1 - e^{-\frac{p_{\perp}^2}{2|eB|}}) \ln \frac{|eB|}{m^2} + e^{-\frac{p_{\perp}^2}{2|eB|}} \int_0^1 du \frac{p_{\parallel}^2 u}{p_{\parallel}^2 u(1-u) + m^2}]}. \quad (66)$$

Then, by using this $D_{\pi}^{(tree)}(p)$ and Eq. (51) for the vertex $\Gamma_{4\pi}$, we find the following one-loop nonplanar contribution to the propagator:

$$\frac{N|eB|}{4\pi^3} \int \frac{d^4 k}{(2\pi)^4} e^{-\frac{p_{\perp}^2 + k_{\perp}^2}{2|eB|}} e^{\frac{i}{eB}(p^1 k^2 - p^2 k^1)} I(p_{\parallel}, k_{\parallel}) D_{\pi}^{(tree)}(k), \quad (67)$$

where

$$I(p_{\parallel}, k_{\parallel}) = \int d^2 l_{\parallel} \frac{(l_{\parallel}^2 + m^2 + l_{\parallel} \cdot p_{\parallel})[(p_{\parallel} - k_{\parallel} + l_{\parallel})^2 + m^2 - (p_{\parallel} - k_{\parallel} + l_{\parallel}) \cdot p_{\parallel}] + p_{\parallel}^2 m^2}{(l_{\parallel}^2 + m^2)[(p_{\parallel} + l_{\parallel})^2 + m^2][(p_{\parallel} - k_{\parallel} + l_{\parallel})^2 + m^2][(l_{\parallel} - k_{\parallel})^2 + m^2]}.$$

Here the integral over transverse momenta k_{\perp} is

$$\int \frac{d^2 k_{\perp}}{(2\pi)^2} e^{-\frac{p_{\perp}^2 + k_{\perp}^2}{2|eB|}} e^{\frac{i}{eB}(p^1 k^2 - p^2 k^1)} D_{\pi}^{(tree)}(k). \quad (68)$$

It is clear that due to the presence of the factor $e^{-k_{\perp}^2/2|eB|}$ and because $D_{\pi}^{(tree)}(k)$ is finite as $k_{\perp}^2 \rightarrow \infty$, this integral is convergent for all values of p_{\perp} , including $p_{\perp} = 0$, and therefore there is no UV/IR mixing in this case. On the other hand, if the factor $e^{-\frac{p_{\perp}^2 + k_{\perp}^2}{2|eB|}}$ were absent in integrand (68), we would get the integral

$$\int \frac{d^2 k_{\perp}}{(2\pi)^2} e^{\frac{i}{eB}(p^1 k^2 - p^2 k^1)} D_{\pi}^{(tree)}(k) \quad (69)$$

which diverges quadratically at $p_{\perp} = 0$, i.e., the UV/IR mixing would occur.

Let us now turn to the description using the smeared fields. The relation (31) between the fields π and Π implies that their propagators are related as

$$D_{\Pi}(p) = e^{\frac{-p_{\perp}^2}{2|eB|}} D_{\pi}(p). \quad (70)$$

Since $e^{\frac{-p_{\perp}^2}{2|eB|}}$ is an entire function, the absence of the UV/IR mixing in the propagator D_{π} implies that there is no UV/IR mixing also in the propagator D_{Π} . This conclusion can be checked directly, by adapting the calculations of the one-loop correction in the propagator D_{π} to the D_{Π} propagator. In this case, it is the form-factor $e^{\frac{-p_{\perp}^2}{2|eB|}}$, built in the propagator $D_{\Pi}^{(tree)}(p)$, that is responsible for the absence of the UV/IR mixing.

This concludes the analysis in 3 + 1 dimensions. In the next Section, we will generalize this analysis to arbitrary dimensions $D = d + 1$ with $d \geq 2$.

V. THE NJL MODEL IN A MAGNETIC FIELD AS A NCFT: BEYOND 3+1 DIMENSIONS

In this Section, we will generalize our analysis to arbitrary dimensions $D = d + 1$ with $d \geq 2$. We begin by considering the NJL model in a magnetic field in 2+1 dimensions, choosing its Lagrangian density similar to that in 3 + 1 dimensions:

$$L = \frac{1}{2}[\bar{\psi}, (i\gamma^{\mu} D_{\mu})\psi] + \frac{G_2}{2N} [(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\psi)^2]. \quad (71)$$

Here a reducible four-dimensional representation of the Dirac matrices is used (for details, see [14]). In a weak coupling regime, the dynamical mass in this model is [14]

$$m = \frac{G_2 |eB|}{2\pi}. \quad (72)$$

The LLL propagator is obtained from the (3+1)-dimensional propagator in Eqs. (20) and (21) by just omitting the x^3 and k^3 variables there:

$$S(x, y) = P(\vec{x}, \vec{y}) S_{||}(x^0 - y^0), \quad (73)$$

where, instead (27), the expression for $S_{||}(x^0 - y^0)$ is:

$$S_{||}(x^0 - y^0) = \int \frac{dk_0}{2\pi} e^{ik_0(x^0 - y^0)} \frac{i}{k_0 \gamma^0 - m} \frac{1 - i\gamma^1 \gamma^2 \text{sign}(eB)}{2}. \quad (74)$$

The analysis now proceeds as in the 3 + 1 dimensional case. The present model corresponds to a noncommutative field theory describing neutral composites σ and π . Its action written through the star product is:

$$\Gamma_2 = \frac{N|eB|}{2\pi} \int d^2x \left[-iTr_{||} [\mathcal{P} \text{Ln} (i\gamma^0 \partial_0 - (\Sigma + i\gamma^5 \Pi))] - \frac{\pi}{G_2 |eB|} \int dx^0 (\sigma^2 + \pi^2) \right]_*, \quad (75)$$

where Σ and Π are smeared fields (compare with Eq. (54)). The action can be also written directly in the noncommutative coordinate space:

$$\Gamma_2 = \frac{N|eB|}{2\pi} \text{Tr} \left[-iTr_{||} [\mathcal{P} \text{Ln} (i\gamma^0 \partial_0 - (\hat{\Sigma} + i\gamma^5 \hat{\Pi}))] - \frac{\pi}{G_2 |eB|} \int dx^0 (\hat{\sigma}^2 + \hat{\pi}^2) \right] \quad (76)$$

(compare with Eq. (55)).

In the previous Sections, it was shown that in the regime with the LLL dominance, the divergences in (3 + 1)-dimensional model are generated only by the (1+1)-dimensional longitudinal dynamics. For the (2 + 1)-dimensional model in this regime, a stronger statement takes place: the model is *finite*. It can be shown by repeating the analysis used in 3 + 1 dimensions. In particular, in the continuum limit $\Lambda \rightarrow \infty$, the effective potential in this model is finite without any renormalizations:

$$V_2(\sigma, \pi) = \frac{N(\sigma^2 + \pi^2)}{2G_2} - \frac{N|eB|\sqrt{\sigma^2 + \pi^2}}{2\pi}. \quad (77)$$

Using Eq. (72), one can express the coupling constant G_2 in the potential through m and $|eB|$. Then the potential takes an especially simple form:

$$V_2(\sigma, \pi) = \frac{N|eB|}{2\pi} \left(\frac{\sigma^2 + \pi^2}{2m} - \sqrt{\sigma^2 + \pi^2} \right). \quad (78)$$

For momenta $k \ll \sqrt{|eB|}$, the dispersion relation for π excitations is [14]

$$E_\pi \simeq \frac{\sqrt{2}m}{|eB|^{1/2}} (\vec{k}^2)^{1/2}. \quad (79)$$

Therefore, as in 3+1 dimensions, the velocity $|\vec{v}| = |\partial E_\pi / \partial \vec{k}|$ is strongly suppressed: in the present case it is of order $m/|eB|^{1/2}$. As to the σ excitation, its "mass", defined as the energy at zero momentum, is very large: $M_\sigma \sim (\sqrt{eB}/m)^{1/2} \sqrt{|eB|}$ [14]. Therefore the σ -mode decouples from the dynamics with $k \ll \sqrt{eB}$.

As in the case of 3+1 dimensions, this truncated [based on the LLL dynamics] model is self-contained and one can consider arbitrary large values for momenta there. It is easy to check that for $k \gg \sqrt{|eB|}$ the dispersion relation for π excitations takes the form

$$E_\pi \simeq m \left(2 - e^{-\frac{\vec{k}^2}{2|eB|}} \right). \quad (80)$$

In this regime, the velocity becomes extremely small, $|\vec{v}| \sim \frac{m|\vec{k}|}{|eB|} e^{-\vec{k}^2/2|eB|}$, and a π excitation is a loosely bound state. Its mass is close to the $2m$ threshold.

As was shown in Section IV, in the limit $|eB| \rightarrow \infty$ the (3+1)-dimensional model is reduced to a continuum set of independent (1+1)-dimensional Gross-Neveu models labeled by the coordinates in the plane perpendicular to the magnetic field. Similarly to that, in the case of $2 + 1$ dimensions, in the limit $|eB| \rightarrow \infty$ the model is reduced to a set of (0+1)-dimensional (i.e., quantum mechanical) models labeled by two spatial coordinates.

A new feature of the (2+1)-dimensional model is a confinement dynamics for charged particles: they do not propagate in a magnetic background. On the other hand, since neutral composites are free to propagate in a magnetic field, one can define asymptotic states and S -matrix for them. The S -matrix should be unitary in the subspace of neutral composites.

Let us now consider the case of higher dimensions $D = d + 1$ with $d > 3$. First of all, recall that for an even d , by using spatial rotations, the noncommutativity tensor θ^{ab} in a noncommutative theory with $[\hat{x}^a, \hat{x}^b] = i\theta^{ab}$ can be reduced to the following canonical skew-diagonal form with skew-eigenvalues θ^a , $a = 1, \dots, \frac{d}{2}$ [1]:

$$\theta^{ab} = \begin{pmatrix} 0 & \theta^1 & & & & \\ -\theta^1 & 0 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & 0 & \theta^{d/2} \\ & & & & -\theta^{d/2} & 0 \end{pmatrix}. \quad (81)$$

If d is odd, then the number of canonical skew-eigenvalues of θ^{ab} is equal to $[\frac{d}{2}]$, where $[\frac{d}{2}]$ is the integer part of $d/2$, and the canonical form of θ^{ab} is similar to (81) except that there are additional one zero column and one zero row.

On the other hand, a constant magnetic field in d dimensions is also characterized by $[\frac{d}{2}]$ independent parameters, and the strength tensor F^{ab} can be also reduced to the canonical skew-diagonal form [16, 24]:

$$F^{ab} = \sum_{c=1}^{[\frac{d}{2}]} B^c (\delta_{2c-1}^a \delta_{2c}^b - \delta_{2c-1}^b \delta_{2c}^a).$$

The corresponding nonzero components of the vector potential are equal to

$$\vec{A}^{ext} = \left(-\frac{B^1 x^2}{2}, \frac{B^1 x^1}{2}, \dots, -\frac{B^{[\frac{d}{2}]} x^{2[\frac{d}{2}]}}{2}, \frac{B^{[\frac{d}{2}]} x^{2[\frac{d}{2}]-1}}{2} \right).$$

Thus, we see that there is one-to-one mapping between the skew-eigenvalues of the noncommutativity tensor θ^{ab} and the independent parameters of the spatial part of the electromagnetic strength tensor F^{ab} in a space of any dimension $d \geq 2$.

Chiral symmetry breaking in the NJL model in a strong magnetic field in dimensions with $d > 3$ was studied in [16]. By using results of that paper, it is not difficult to extend our analysis in 3+1 and 2+1 dimensions to the case of $d > 3$. The crucial point in the analysis is the structure of the Fourier transform of the translationally invariant part of the LLL propagator. If all B^a are nonzero, one can show that it is

$$\tilde{S}_{[\frac{d}{2}]}(k) = i \exp \left[- \sum_{a=1}^{[\frac{d}{2}]} \frac{k_{2a-1}^2 + k_{2a}^2}{|eB^a|} \right] \frac{k_{||} \gamma_{||} + m}{k_{||}^2 - m^2} \Pi_{a=1}^{[\frac{d}{2}]} (1 - i \gamma^{2a-1} \gamma^{2a} \text{sign}(eB^a)), \quad (82)$$

where $k_{||} = k^0$ if d is even and $k_{||} = (k^0, k^d)$ if d is odd. If some $B^c = 0$, then, for each c , the longitudinal part $k_{||}$ gets two additional components, k^{2c-1} and k^{2c} , and the corresponding terms are absent in the transverse part of expression (82). Thus, like in 3+1 and 2+1 dimensions, the LLL propagator factorizes into the transverse and longitudinal parts. The projection operator $P_n(x_{\perp}, y_{\perp})$ on the LLL is now equal to the direct product of the projection operators (26) in the $x^{2a-1} x^{2a}$ -planes with nonzero B^a [here the subscript n is the number of nonzero independent components of F^{ab}].

Because of that, it is clear that the NJL model in a strong magnetic field in a space of arbitrary dimensions $d \geq 2$ corresponds to a noncommutative field theory with parameters θ^{ab} expressed through the magnetic part of the strength tensor F^{ab} . Its action is [compare with expressions (55) and (76)]:

$$\Gamma_n = N \text{Tr} \left[- \frac{i \Pi_{a=1}^n |eB^a|}{(2\pi)^n} \text{Tr}_{||} \left[\mathcal{P}_n \text{Ln} \left(i \gamma_{||} \partial_{||} - (\hat{\Sigma} + i \gamma^5 \hat{\Pi}) \right) \right] - \frac{1}{2G_d} \int d^{D-2n} x_{||} (\hat{\sigma}^2 + \hat{\pi}^2) \right], \quad (83)$$

where n is the number of nonzero independent components of F^{ab} and the projector \mathcal{P}_n equals the direct product of projectors (53) in the $x^{2a-1}x^{2a}$ -planes with nonzero B^a . In particular, for a magnetic field configuration with the maximal number $n = [d/2]$ of independent nonzero tensor components, the dynamics is quasi-(1+1)-dimensional for odd d and finite for even d . In the latter case the model describes a confinement dynamics of charged particles. Also, as all $|eB^a| \rightarrow \infty$, the model is reduced either to a continuum set of (1+1)-dimensional GN models labeled by $d - 1$ spatial coordinates (odd d) or to a set of quantum mechanical models labeled by d spatial coordinates (even d).

VI. CONCLUSION

The main result of this paper is that in any dimension $D = d + 1$ with $d \geq 2$, the NJL model in a strong magnetic field determines a consistent NCFT. These NCFT are quite sophisticated that is reflected in their action (83) expressed through the smeared fields Σ and Π with built-in exponentially damping form-factors. These form-factors occur in the propagators of the smeared fields and are responsible for removing the UV/IR mixing that plagues conventional nonsupersymmetric NCFT [10]. As an alternative, one can also use the composites fields σ and π . In this case, the form-factors occur in their interaction vertices and this again leads to the removal of the UV/IR mixing.

An especially interesting case is that for a magnetic field configuration with the maximal number $[d/2]$ of independent nonzero tensor components. In that case, the dynamics is quasi-(1+1)-dimensional for odd d and finite for even d . How can it be, despite the fact that the initial NJL model is nonrenormalizable for $d \geq 2$? And, moreover, how can it happen in theories in which neutral composites propagate in a bulk of a space of *arbitrary* high dimensions? The answer to these questions is straightforward. The initial NJL model in a strong magnetic field and the truncated model based on the LLL dynamics are essentially identical only in infrared, with momenta $k \ll \sqrt{|eB|}$. At large momenta, $k \gg \sqrt{|eB|}$, these two models are very different. It is the LLL dominance that provides the exponentially damping (form-)factors which are responsible for finiteness of the present model for even d and its quasi-(1+1)-dimensional character for odd d . Thus, besides being a low energy theory of the NJL model in a strong magnetic field, the NCFT based on the LLL dynamics is self-contained and self-consistent.

As was discussed in Section II, the exponentially damping factors occur also in nonrelativistic quantum mechanical models. In particular, they are an important ingredient of the formalism of the projection onto the LLL developed for studies of condensed matter systems in Refs. [12, 13]. It is then natural to ask why do such factors not appear also in string theories in a magnetic field? The answer to this question is connected with a completely different way that open strings respond to a strong B field. It can be seen already on the classical level. Indeed, due to the boundary conditions at the ends of open strings, their length *grows* with B until the string tension compensates the Lorentz forces exerted at the ends of strings [9]. In contrast to that, in quantum field and condensed matter systems, charged particles, which form neutral composites, move along circular orbits in a magnetic field, and their radius *shrinks* with increasing B . This leads to the Landau type wave functions of composites and, therefore, to the exponentially damping (form-)factors either in vertices (for σ and π fields) or in propagators (for smeared fields).

Therefore, unlike the dynamics of neutral composites in condensed matter and quantum field systems, open strings in a magnetic background do lead to the conventional NCFT. Since these theories are supersymmetric, the UV/IR mixing affects only the constants of renormalizations and does not destroy their consistency [26]. Thus different physical systems in a magnetic fields lead to different classes of consistent NCFT.³

In the present paper, as an example, we considered the NJL model in a magnetic field. It is however clear that because of the universality of the dynamics connected with transverse coordinates, form-factors should also occur in propagators (or in vertices) in more complicated field theories in a magnetic field (although the form of the form-factors can vary). Therefore the corresponding NCFT should be in this regard similar to those revealed in this paper.

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³ In this regard, quantum mechanical systems in a strong magnetic field are special. As was shown in Section II, depending on two different treatments of the case with the mass $m = 0$, they determine either conventional NCFT, as it is done in Ref. [3], or NCFT with exponentially damping form-factors.

APPENDIX A

In this Appendix, we will show that in the case of fields independent of the longitudinal coordinates $x_{||}$, all their interaction vertices $\Gamma_{n\phi}$ ($n \geq 3$) can be rewritten through the star product.

The relevant part of the n -point vertex $\Gamma_{n\phi}$ is the part which includes the integration over transverse coordinates. It has the form:

$$\Gamma_{n\phi}^\perp \equiv \int d^2x_1 \dots d^2x_n P(x_1, x_2) \phi(x_2) P(x_2, x_3) \phi(x_3) \dots P(x_n, x_1) \phi(x_1), \quad (84)$$

where $P(x_1, x_2)$ is the transverse part of the fermion propagator written in Eq. (26) [here, for convenience, we omitted the subscript \perp in transverse coordinates].

Expressing the fields ϕ through their Fourier transforms, one can explicitly integrate over x_i coordinates in (84) [the integrals are Gaussian]. It can be done step by step. First, we find

$$I_1(x_1, x_3) = \int d^2x_2 P(x_1, x_2) e^{i\vec{k}_2 \vec{x}_2} P(x_2, x_3) = P(x_1, x_3) e^{-\frac{\vec{k}_2^2}{2|eB|}} e^{\frac{\text{sign}(eB) \epsilon^{ab} k_2^a (x_1 - x_3)^b}{2}} e^{\frac{i}{2} \vec{k}_2 (\vec{x}_1 + \vec{x}_3)}. \quad (85)$$

The second step leads to an expression with a similar structure:

$$I_2(x_1, x_4) = \int d^2x_3 I_1(x_1, x_3) e^{i\vec{k}_3 \vec{x}_3} P(x_3, x_4) = P(x_1, x_4) e^{-\frac{\vec{k}_2^2 + \vec{k}_3^2 + \vec{k}_2 \vec{k}_3}{2|eB|}} e^{-\frac{i}{2eB} \epsilon^{ab} k_2^a k_3^b} e^{\frac{\text{sign}(eB) \epsilon^{ab} (k_2 + k_3)^a (x_1 - x_4)^b}{2}} e^{\frac{i}{2} (\vec{k}_2 + \vec{k}_3) (\vec{x}_1 + \vec{x}_4)}. \quad (86)$$

Proceeding in this way until the integration over x_n , we encounter the integral

$$I_{n-1}(x_1, x_1) = \int d^2x_n I_{n-2}(x_1, x_n) e^{i\vec{k}_n \vec{x}_n} P(x_n, x_1). \quad (87)$$

It closes the fermion loop because the last argument in $P(x_n, x_1)$ coincides with the first argument of I_{n-2} . Because of that, the result of this integration is especially simple:

$$I_{n-1}(x_1, x_1) = \frac{|eB|}{2\pi} e^{-\frac{\sum_{i=2}^n \vec{k}_i^2 + \sum_{2 \leq i < j} \vec{k}_i \vec{k}_j}{2|eB|}} e^{-\frac{i}{2eB} (\sum_{2 \leq i < j} \epsilon^{ab} k_i^a k_j^b)} e^{i(\sum_{i=2}^n \vec{k}_i \vec{x}_1)}, \quad (88)$$

where the equality $P(x_1, x_1) = |eB|/2\pi$ was used. The last integration over x_1 yields

$$I_n = \int d^2x_1 I_{n-1}(x_1, x_1) e^{i\vec{k}_1 \vec{x}_1} = 2\pi |eB| \delta^2(\sum_{i=1}^n \vec{k}_i) e^{-\frac{\sum_{i=2}^n \vec{k}_i^2 + \sum_{2 \leq i < j} \vec{k}_i \vec{k}_j}{2|eB|}} e^{-\frac{i}{2eB} (\sum_{2 \leq i < j} \epsilon^{ab} k_i^a k_j^b)}. \quad (89)$$

Here the delta function ensures the conservation of the total transverse momentum. Now, because of the identity

$$\sum_{i=2}^n \vec{k}_i^2 + \sum_{2 \leq i < j} \vec{k}_i \vec{k}_j = - \sum_{1 \leq i < j} \vec{k}_i \vec{k}_j + (\sum_{i=1}^n \vec{k}_i)^2 - \vec{k}_1 (\sum_{i=1}^n \vec{k}_i)$$

and the conservation of the total momentum, we obtain the equalities

$$\sum_{i=2}^n \vec{k}_i^2 + \sum_{2 \leq i < j} \vec{k}_i \vec{k}_j = - \sum_{1 \leq i < j} \vec{k}_i \vec{k}_j = \frac{1}{2} \sum_{i=1}^n \vec{k}_i^2$$

and $\sum_{2 \leq i < j} \epsilon^{ab} k_i^a k_j^b = \sum_{1 \leq i < j} \epsilon^{ab} k_i^a k_j^b$. Using these equalities, we conclude that the exponential term in expression (89) can be rewritten through the cross product as $e^{-\frac{\sum_{i=1}^n \vec{k}_i^2}{4|eB|}} e^{-\frac{i}{2} \sum_{i < j} k_i \times k_j}$. Therefore, similarly to three and four point vertices (32) and (33), a generic n -point interaction vertex $\Gamma_{n\Phi}$ ($n \geq 3$) has the following structure:

$$\Gamma_{n\Phi} = C_n \frac{N|eB|}{m^{n-2}} \int d^2x_{||} \frac{d^2k_1 \dots d^2k_n}{(2\pi)^{2n}} \Phi(k_1) \dots \Phi(k_n) \delta^2(\sum_i k_i) e^{-\frac{i}{2} \sum_{i < j} k_i \times k_j}, \quad (90)$$

where here Φ represents the smeared fields Π and $\tilde{\Sigma}$ and C_n is a numerical constant which can be easily found by expanding the effective potential in the Taylor series in constant fields π and $\tilde{\sigma}$. Equation (90) in turn implies that the vertex $\Gamma_{n\Phi}$ can be rewritten through the star product in the coordinate space as

$$\Gamma_{n\Phi} = C_n \frac{N|eB|}{4\pi^2 m^{n-2}} \int d^2 x_{||} d^2 x_{\perp} \Phi_1 * \Phi_2 * \dots * \Phi_n \quad (91)$$

(compare with expressions in Eq. (35)). In the noncommutative coordinate space, the vertex is:

$$\Gamma_{n\Phi} = C_n \frac{N|eB|}{4\pi^2 m^{n-2}} \text{Tr} \hat{\Phi}_1 \hat{\Phi}_2 \dots \hat{\Phi}_n \quad (92)$$

(compare with Eq. (40)).

APPENDIX B

In this Appendix, it will be shown that the exponentially damping factors and the M -star product naturally appear in the formalism of the projected density operators on the LLL states developed in studies of the quantum Hall effect in Ref. [21]. To be concrete, we will consider the $\Gamma_{4\pi}$ vertex in this formalism.

As follows from Eq. (16), the $\Gamma_{4\pi}$ vertex is given by

$$\Gamma_{4\pi} = \frac{i}{4} \int d^4 x d^4 y d^4 z d^4 v \text{tr} [S(x, y) \gamma^5 \pi(y) S(y, z) \gamma^5 \pi(z) S(z, v) \gamma^5 \pi(v) S(v, x) \gamma^5 \pi(x)] . \quad (93)$$

According to Eq. (25), the dependence on the transverse x_{\perp} and longitudinal $x_{||}$ coordinates factorizes in the LLL propagator $S(x, y)$. If fields π in (93) do not depend on $x_{||}$, then it is straightforward to integrate over the longitudinal coordinates in this expression that yields the factor

$$\frac{i}{4} \int \frac{d^2 k_{||}}{(2\pi)^2} \text{tr} \left(\frac{1}{k_{||} \gamma_{||} - m} \frac{1 - i \gamma^1 \gamma^2 \text{sign}(eB)}{2} \gamma^5 \right)^4 = -\frac{1}{8\pi m^2}. \quad (94)$$

To get the $\Gamma_{4\pi}$ vertex, we now need to calculate the transverse part

$$\int d^2 x_{\perp} d^2 y_{\perp} d^2 z_{\perp} d^2 v_{\perp} P(x_{\perp}, y_{\perp}) \pi(y_{\perp}) P(y_{\perp}, z_{\perp}) \pi(z_{\perp}) P(z_{\perp}, v_{\perp}) \pi(v_{\perp}) P(v_{\perp}, x_{\perp}) \pi(x_{\perp}). \quad (95)$$

We will use the formalism of projected density operators [21] to calculate it. The crucial point is the fact that the transverse part of the LLL fermion propagator $P(x, y)$ is the the projection operator on the LLL states (henceforth we will omit the subscript \perp for the transverse coordinates). Namely,

$$P(x, y) = \sum_n \langle x | n \rangle \langle n | y \rangle, \quad (96)$$

where the sum is taken over all LLL states, which in the symmetric gauge are

$$\psi_n(z, z^*) = \left(\frac{|eB|}{2} \right)^{\frac{n+1}{2}} \frac{z^n}{\sqrt{\pi n!}} e^{-\frac{|eB|}{4} z z^*} \quad (97)$$

with $z = x^1 - i \text{sign}(eB) x^2$. Now, by using completeness relations like

$$\int d^2 y \langle n_1 | y \rangle \pi(y) \langle y | n_2 \rangle = \langle n_1 | \pi | n_2 \rangle,$$

we obtain expression (95) in the form

$$\sum_{n_1, \dots, n_4} \langle n_1 | \pi | n_2 \rangle \dots \langle n_3 | \pi | n_4 \rangle. \quad (98)$$

To get the $\Gamma_{4\pi}$ interaction vertex in the momentum space, we will use the Fourier transforms of fields π . Then we encounter factors of the form:

$$\langle n_i | \rho_k | n_j \rangle,$$

where $\rho_k = e^{i\vec{k}\vec{x}} = \exp\left[\frac{i}{2}(kz^* + k^*z)\right]$, with $k = k^1 - i \text{sign}(eB)k^2$, is called the density operator.

In what follows, we will use the methods developed in Refs. [12, 13] and, in fact, follow very closely Ref. [25]. First of all, since the prefactor in expression (97) is analytic in z , the factor $e^{\frac{i}{2}k^*z}$ in ρ_k acts entirely within the LLL. On the other hand, another factor $e^{\frac{i}{2}kz^*}$ in ρ_k contains z^* and therefore does not act within the LLL. Actually, the following relation takes place:

$$\langle n|(z^*)^s|m\rangle = \langle n|\left(\frac{2}{|eB|}\frac{\partial}{\partial z} + \frac{z^*}{2}\right)^s|m\rangle, \quad (99)$$

which expresses the matrix elements of z^* between the LLL states in terms of the operator $\hat{z} = \frac{2}{|eB|}\frac{\partial}{\partial z} + \frac{z^*}{2}$ that acts within the LLL. Therefore, on the LLL states, we can replace the density operator $\exp\left[\frac{i}{2}(kz^* + k^*z)\right]$ by the projected density operator $\hat{\rho}_k = e^{\frac{i}{2}k\hat{z}}e^{\frac{i}{2}k^*z}$.

Now, using the the projected density operators $\hat{\rho}_k$ for the $\Gamma_{4\pi}$ vertex, we get:

$$\Gamma_{4\pi} = -\frac{1}{8\pi m^2} \int d^2x_{||} \int \frac{d^2k_1 d^2k_2 d^2k_3 d^2k_4}{(2\pi)^8} \pi(k_1)\pi(k_2)\pi(k_3)\pi(k_4) \sum_{n_1, \dots, n_4} (\hat{\rho}_{k_1})_{n_1 n_2} (\hat{\rho}_{k_2})_{n_2 n_3} (\hat{\rho}_{k_3})_{n_3 n_4} (\hat{\rho}_{k_4})_{n_4 n_1}. \quad (100)$$

Since the LLL states form a complete basis for the operators $\hat{\rho}_k$, we have

$$\sum_{n_2} (\hat{\rho}_{k_1})_{n_1 n_2} (\hat{\rho}_{k_2})_{n_2 n_3} = (\hat{\rho}_{k_1} \hat{\rho}_{k_2})_{n_1 n_3}.$$

The product of two projected density operators is given by [25]

$$\hat{\rho}_{k_1} \hat{\rho}_{k_2} = \exp\left[\frac{\vec{k}_1 \vec{k}_2}{2|eB|} - \frac{i}{2}k_1 \times k_2\right] \hat{\rho}_{k_1+k_2}. \quad (101)$$

Notice that the exponent in this equation can be rewritten through the M -cross product (41) as $e^{-\frac{i}{2}k_1 \times_M k_2}$.

Therefore, we find the following expression for $\Gamma_{4\pi}$:

$$\Gamma_{4\pi} = -\frac{1}{8\pi m^2} \int d^2x_{||} \int \frac{d^2k_1 d^2k_2 d^2k_3 d^2k_4}{(2\pi)^8} \pi(k_1)\pi(k_2)\pi(k_3)\pi(k_4) e^{-\frac{i}{2}\sum_{i<j} k_i \times_M k_j} \sum_n (\hat{\rho}_{k_1+k_2+k_3+k_4})_{nn}. \quad (102)$$

Using further the relation (see Ref. [21])

$$\sum_n (\hat{\rho}_{k_1+k_2+k_3+k_4})_{nn} = N \delta_{\sum_i \vec{k}_i, 0},$$

where $N = S \frac{|eB|}{2\pi}$ is the number of states on the LLL and S is the square of the transverse plane, and the identity

$$S \delta_{\sum_i \vec{k}_i, 0} = (2\pi)^2 \delta^2\left(\sum_i \vec{k}_i\right),$$

we finally get the expression for the vertex $\Gamma_{4\pi}$ that coincides with expression (45).

Thus we see that the mathematical reason for the appearance of exponentially damping factors and the M -star product is related to the algebra of the projected density operators (101). Obviously, the generalization of the above calculations to an arbitrary interaction vertex for the π and $\tilde{\sigma}$ fields is straightforward.

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